KMA315 Analysis 3A
Lecture Notes

UNIVERSITY of TASMANIA

Notes authored by Nicholas Ham

Semester 1, 2016

"Analysis is where we know what everything is, but have very little idea of how anything works, which is somewhat opposed to algebra where we know how everything works, but have very little idea of what anything is."

Preface

These notes have been typed for the real analysis component of Analysis 3A, a unit through the School of Physical Sciences at the University of Tasmania.

On top of attempting to cover the material for the real analysis component, they have been written in a manner that aims to introduce students to:

(i) proof-based mathematics;

(ii) reading and writing mathematics, both within and outside the scope of the unit;

(iii) using the internet and other common resources to extend their understanding and knowledge both during and proceeding the unit, with the intention being (comparing with a very well-known analogy) to teach students the skills to fish over giving them a platter of fish;

(iv) considering their own intended audience when writing mathematics for themselves, especially as one's audience grows; and

(v) considering whether they are the intended audience when reading mathematics and attending mathematics talks, and how to make the most out of situations where their knowledge-base is either below or above the intended audience.

Contents

Notation

Below is a (hopefully comprehensive) table of notation used throughout these notes.

$\in$	is an element of	
$\notin$	is <u>not</u> an element of	
$\subseteq$	is a subset of	
$\not\subseteq$	is <u>not</u> a subset of	
$\exists$	there exists	
$\nexists$	there does not exist	
$\forall$	for all	
iff	if and only if	
Eg.	for example	
Ie.	that is	
$\mathbb{Z}$	integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}_+$	positive integers	$\{1, 2, 3, \ldots\}$
$\mathbb{Z}_{\geq 0}$	non-negative integers	$\{0, 1, 2, \ldots\}$
$\mathbb{Q}$	rational numbers	$\left\{\frac{a}{b} : a, b \in \mathbb{Z}\right\}$
$\mathcal{C}(\mathbb{Q})$	irrational numbers	
$\mathbb{R}$	real numbers	$\mathbb{Q} \cup \mathcal{C}(\mathbb{Q})$
$\mathbb{R}_+$	positive real numbers	$\{r \in \mathbb{R} : r > 0\}$
$\mathbb{R}\backslash\{0\}$	non-zero real numbers	$\{r \in \mathbb{R} : r \neq 0\}$
$\mathbb{C}$	complex numbers	$\{a + bi : a, b \in \mathbb{R}\}$

$X \times Y$	Cartesian product (ordered pairs) of X and Y	$\{(x, y) : x \in X, y \in Y\}$
$X^2 = X \times X$		
$X \times Y \times Z$	Cartesian product of X, Y and Z	$\{(x, y, z) : x \in X, y \in Y, z \in Z\}$
$\min A$	minimum element of A	
$\max A$	maximum element of A	
$\sup A$	least upper bound of A	
$\inf A$	greatest lower bound of A	
$(y_n)_{n=0}^{\infty}$	sequence of real numbers	
$\lim_{n\to\infty} y_n$	limit of $(y_n)_{n=0}^{\infty}$ (when it exists)	
$f : X \to Y$	Y-valued function on X	

Chapter 0

Notation, terminology & proofs

0.1 Set notation

Straight from Wikipedia

> "In mathematics, a set is a collection of distinct objects, considered as an object in its own right. For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively they form a single set of size three, written $\{2, 4, 6\}$." `https://en.wikipedia.org/wiki/Set_(mathematics)`

0.1.1 Definition: The objects of a set are also typically referred to as *elements*. Given an object x and a set S:

(i) if x is an element of a set S, then the relationship is typically denoted by $x \in S$; whereas

(ii) if x is not an element of S, then the relationship is typically denoted by $x \notin S$.

0.1.2 Examples: (non)-examples of set elements include:

(i) $3 \in \{2, 3, 4\}$;

(ii) $5 \notin \{2, 3, 4\}$;

(iii) $\{1,2\} \in \{\{1\},\{1,2\},\{1,2,3\}\}$; and

(iv) $\{1,3\} \notin \{\{1\},\{1,2\},\{1,2,3\}\}$.

0.1.3 Definition: Let X and Y be sets. When every element of X is an element of Y, it is typically said that X *is a subset of* Y and denoted by $X \subseteq Y$, whereas if there exists $x \in X$ such that $x \notin Y$, then it is typically said that X *is not a subset of* Y and denoted by $X \not\subseteq Y$.

0.1.4 Examples: (non)-examples of subsets are:

(i) $\{2,3\} \subseteq \{1,2,3,4\}$;

(ii) $\{1,3,5\} \not\subseteq \{1,2,3,4\}$ (since $5 \notin \{1,2,3,4\}$);

(iii) $\{1,2\} \not\subseteq \{\{1\},\{2\}\}$ (since $1 \notin \{\{1\},\{2\}\}$ and since $2 \notin \{\{1\},\{2\}\}$); and

(iv) $\{1,2\} \not\subseteq \{\{1\},\{1,2\},\{1,2,3\}\}$.

0.1.1 Describing sets

When the elements of a set X are listed exhaustively, it is often referred to as *the extensional definition of* X. We may also describe sets using what is referred to as *an intensional set definition*, where we start with a larger set then restrict the elements.

0.1.5 Examples: (i) The set of integers x such that x is greater than 5; and

(ii) The set of real numbers x such that x is a root of $ax^2 + bx + c$.

The intensional set definitions from Examples 0.1.5 may be denoted as follows:

(i) $\{x \in \mathbb{Z} : x > 5\}$; and

(ii) $\left\{x \in \mathbb{R} : x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right\}$.

Note that the terminology *intensional* and *extensional* is not necessary to retain in one's memory, or that commonly used (at least in the author's experience), however the ability to correctly interpret sets described in either form is incredibly important.

0.2 Number sets ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, etc.)

0.2.1 Definition: We typically denote by:

(i) $\mathbb{Z}$ the set of integers, ie. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$;

(ii) $\mathbb{N}$ the set of natural numbers, ie. $\mathbb{N} = \{0, 1, 2, \ldots\}$;

(iii) $\mathbb{Q}$ the set of rational numbers, ie. $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}\}$;

(iv) $\mathbb{R}$ the set of real numbers (which includes the rational and irrational numbers); and

(v) $\mathbb{C}$ the set of complex numbers, ie. $\{a + bi : a, b \in \mathbb{R}\}$.

We will further denote by (note that the following notation is reasonably common, however it is also not uncommon for it to differ):

(i) $\mathbb{Z}_+$ the set of positive integers, ie. $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$;

(ii) $\mathbb{Z}_{\geq 0}$ the set of non-negative integers, ie. $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$;

(iii) $\mathcal{C}(\mathbb{Q})$ the set of irrational numbers, ie. numbers whose decimal expansion has an infinite number of non-zero values;

(iv) $\mathbb{R}_+$ the set of positive real numbers, ie. $\mathbb{R}_+ = \{r \in \mathbb{R} : r > 0\}$; and

(v) $\mathbb{R}\backslash\{0\}$ the set of non-zero real numbers, ie. $\mathbb{R}\backslash\{0\} = \{r \in \mathbb{R} : r \neq 0\}$.

0.3 Using terminology efficiently and effectively

One might consider there to be two camps when it comes to choosing terminology, those who prefer to attribute terminology to famous people (usually academics) who may or may not have actually had anything to do with what they are naming, and those who prefer to use terminology which is unambiguous and somewhat descriptive of what it is naming.

The reader ought to be able to guess which camp the author would self-select himself into based on the phrasing of the previous paragraph, however in case it is not clear, yes the author prefers notation and terminology to be descriptive and unambiguous.

0.4 Proof-based mathematics

0.5 Understanding proofs

Very loosely speaking, a proof is a sequence of logically-valid steps which takes us from one statement to another. Some tips for reading proofs, in no particular order of importance, are:

- Consider who the author may have considered the intended reader(s) to be, especially whether your level of mathematical maturity is: that of an intended reader; below that of an intended reader; or above that of an intended reader; and
- Different people have different writing styles that work (or don't) for them, try to get a feel for what the writer's (or writers') style is and how you can make the most out of it, even in situations where you need to read something which isn't in one of the styles you prefer. One could argue that one aspect of mathematical maturity is the ability to properly digest information from a range of different writing styles.

0.6 Writing proofs

Here are some tips when writing proofs, in no particular order of importance:

- Consider who your reader(s) will be, what their level of mathematical maturity is likely to be, whether you are able to jump steps while being confident it wont lose your reader, or (if you care) whether any readers may question if you (the author) were able to fill the missing logical steps.

Chapter 1

Real Numbers ($\mathbb{R}$)

1.1 Algebraic Properties

1.1.1 Binary Operations

1.1.1 Definition: Given a set X, a *binary operation* on X is a function $* : X \times X \to X$. Though, given say $x, y \in X$, instead of the usual function notation of $*(x, y)$, people will typically use *infix notation* and denote it as $x * y$.

1.1.2 Example: $+$, $-$, $*$, $/$, ^ are all binary operations on real numbers.

1.1.3 Definition: A binary operation $* : X \times X \to X$ is referred to as:

(i) *commutative* if $a * b = b * a$ for all $a, b \in X$; and

(ii) *associative* if $a(bc) = (ab)c$ for all $a, b, c \in X$.

Furthermore:

(i) an element $e \in X$ is referred to as *the identity of* $*$ if $e * x = x = x * e$ for all $x \in X$; and

(ii) given $x \in X$, if there exists $x^{-1} \in X$ such that $x * x^{-1} = e = x^{-1} * x$ then x^{-1} is referred to as *the inverse of* x, and it is said that x *is invertible*.

1.1.4 Proposition: A binary operation $* : X \times X \to X$ can have at most one identity.

Proof. Suppose there exists $e, f \in X$ such that both e and f are identities. Then $e = e * f = f$. $\square$

1.1.5 Proposition: If an element of a set equipped with a binary operation has an inverse, that inverse must be unique.

Proof. Let $x \in X$ and suppose $y, z \in X$ are both inverses of x. Then $y = ye = y(xz) = (yx)z = ez = z$. $\square$

(i) Properties of Addition:

  (I) $a + b = b + a$ for all $a, b \in \mathbb{R}$ (+ is commutative);

  (II) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$ (+ is associative);

  (III) $0 + a = a = a + 0$ for all $a \in \mathbb{R}$ ($\mathbb{R}$ has an additive identity, namely 0);

  (IV) For each $a \in \mathbb{R}$, $-a \in \mathbb{R}$ and $a + (-a) = 0 = (-a) + a$ (each real number has an additive inverse),

(ii) Multiplicative Properties:

  (I) $ab = ba$ for all $a, b \in \mathbb{R}$ (. is commutative);

  (II) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$ (. is associative);

  (III) $1a = a = a1$ for all $a \in \mathbb{R}$ ($\mathbb{R}$ has a multiplicative identity, namely 1);

  (IV) for each $a \in \mathbb{R}\backslash\{0\}$, $\frac{1}{a} \in \mathbb{R}$ and $a\frac{1}{a} = 1 = \frac{1}{a}a$ (each non-zero real number has a multiplicative inverse), and

(iii) Distributive Properties:

  (I) $\begin{cases} a(b + c) = ab + ac \\ (b + c)a = ba + ca \end{cases}$ for all $a, b, c \in \mathbb{R}$ (multiplication distributes over addition).

Note that the above additive, multiplicative and distributive properties of real numbers establish that the real numbers are what is referred to in mathematics as a *field*.

1.2 Intervals of real numbers

1.2.1 Definition: A subset $A \subseteq \mathbb{R}$ is referred to as *an interval of* $\mathbb{R}$ if for each $a, b \in A$, $(a, b) \subseteq A$. A square bracket is typically used to denote the inclusion of an end-point while a round bracket is typically used to denote the exclusion of an end-point. For example:

(i) (a, b) denotes $\{r \in \mathbb{R} : a < r < b\}$;

(ii) $(a, b]$ denotes $\{r \in \mathbb{R} : a < r \leq b\}$;

(iii) $[a, b)$ denotes $\{r \in \mathbb{R} : a \leq r < b\}$; and

(iv) $[a, b]$ denotes $\{r \in \mathbb{R} : a \leq r \leq b\}$;

You may find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Interval_(mathematics)`.

1.3 Supremum and infimum properties (aka least/greatest-upper-bound properties)

1.3.1 Definition: Given $S \subseteq \mathbb{R}$, $x \in \mathbb{R}$ is typically referred to as:

(i) *an upper bound* if for each $s \in S$, $s < x$;

(ii) *a lower bound* if for each $s \in S$, $x < s$;

(iii) *the maximum element in* S if:

  (I) $x \in S$; and

  (II) $y \leq x$ for all $y \in S$.

(iv) *the minimum element in* S if:

  (I) $x \in S$; and

(II) $x \leq y$ for all $y \in S$.

(v) *the least upper bound of* S (aka *the supremum of* S) if:

(I) x is an upper bound of S; and

(II) $x \leq y$ for every upper bound y of S.

(vi) *the greatest lower bound of* S (aka *the infimum of* S) if:

(I) x is a lower bound of S; and

(II) $y \leq x$ for every lower bound y of S.

1.3.2 Example: Let $S = 2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ (Ie. the even non-negative integers).

(i) Every real number in $(-\infty, 0]$ is a lower bound of S;

(ii) S does not have any upper bounds;

(iii) 0 is the minimum element in S;

(iv) There is no maximum element in S;

(v) The greatest lower bound of S is 0; and

(vi) There is no least upper bound of S.

1.3.3 Property: (Least-Upper-Bound/Supremum Property) If A is a non-empty subset of $\mathbb{R}$ that is bounded above then A has a least upper bound in $\mathbb{R}$ (ie. there exists $x \in \mathbb{R}$ such that x is the least upper bound of A).

1.3.4 Property: (Greatest-Lower-Bound/Infimum Property) If A is a non-empty subset of $\mathbb{R}$ that is bounded below then A has a greatest lower bound in $\mathbb{R}$ (ie. there exists $x \in \mathbb{R}$ such that x is the greatest lower bound of A).

1.3.5 Example: Let $S = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$.

(i) Every real number in $(-\infty, 0]$ is a lower bound of S;

(ii) Every real number in $[1, \infty)$ is an upper bound of S;

(iii) 0 is the minimum element in S;

(iv) There is no maximum element in S;

(v) The greatest lower bound of S is 0; and

(vi) The least upper bound of S is 1.

You may find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Least-upper-bound_property`.

1.4 Archimedean Property

1.4.1 Property: For each $a, b \in \mathbb{R}_+$ there exists $n \in \mathbb{Z}_+$ such that $na > b$.

You may find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Archimedean_property`.

1.5 Order properties (non-examinable)

Note: Section 1.5 is fairly self contained, it can be understood without too much knowledge from the remainder of these notes, nor do the remainder of these notes refer back to this section. You can safely skip this section when studying for the exam.

1.5.1 Binary relations

1.5.1 Definition: Given a set X, a *binary relation* on X is a subset of $X \times X$. Given a binary relation $\mathcal{R} \subseteq X \times X$, if $(x, y) \in \mathcal{R}$ it is not uncommon to just write $x\mathcal{R}y$. (Note that binary relations can also be defined more generally as a subset of the Cartesian product of two (possibly) distinct sets).

1.5.2 Examples: The following are all binary relations on the real numbers:

(i) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$;

(ii) $\{(x, y) \in \mathbb{R}^2 : x \leq y\}$;

(iii) $\{(x, y) \in \mathbb{R}^2 : x > y\}$;

(iv) $\{(x, y) \in \mathbb{R}^2 : x \geq y\}$; and

(v) $\{(x, y) \in \mathbb{R}^2 : x = y\}$.

1.5.3 Definition: (Properties of Binary Relations) A binary relation $\mathcal{R} \subseteq X^2$ is referred to as:

(i) *reflexive* when for all $x \in X$, $(x, x) \in \mathcal{R}$;

(ii) *irreflexive* when for all $x \in X$, $(x, x) \notin \mathcal{R}$;

(iii) *symmetric* when for all $x, y \in X$, $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$;

(iv) *antisymmetric* when for all $x, y \in X$, $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies $x = y$;

(v) *transitive* when for all $x, y, z \in X$, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ then $(x, z) \in \mathcal{R}$; and

(vi) *total* when for all $x, y \in X$, either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$ (or both).

1.5.4 Examples: When specifically taking well-known binary relations on real numbers:

(i) $<$ and $>$ are both irreflexive and transitive;

(ii) $\leq$ and $\geq$ are both reflexive, antisymmetric and transitive; and

(iii) $=$ is reflexive, symmetric and transitive.

1.5.2 Orders and equivalence relations

1.5.5 Definition: (Special Types of Binary Relations) A binary relation $\mathcal{R} \subseteq X^2$ is referred to as:

(i) *a pre-order* when it is reflexive and transitive;

(ii) *a partial order* when it is reflexive, antisymmetric and transitive;

(iii) *a total order* when it is a total partial order; and

(iv) *an equivalence relation* when it is reflexive, symmetric and transitive.

1.5.6 Examples: When specifically taking well-known binary relations on real numbers:

(i) $\leq$ and $\geq$ are partial orders; and

(ii) $=$ is an equivalence relation.

1.5.7 Definition: A field $(\mathbb{F}, +, .)$ together with a total order $\leq$ on $\mathbb{F}$ is referred to as *an ordered field* if it satisfies the following properties for all $a, b, c \in \mathbb{F}$:

(i) if $a \leq b$ then $a + c \leq b + c$; and

(ii) if $0 \leq a$ and $0 \leq b$ then $0 \leq ab$.

For some more information about ordered fields, see `http://people.reed.edu/~mayer/math112.html/html1/node19.html`. Particularly of note is that squares must be non-negative in an ordered field.

1.5.3 Ordering $\mathbb{R}$

We already pointed out in Section 1.1 that $(\mathbb{R}, +, .)$ forms a field. It is not difficult to convince oneself that $(\mathbb{R}, +, .)$ together with $\leq$ forms an ordered field. Ie. $\mathbb{R}$ is totally ordered by $\leq$ and satisfy for all $x, y, z \in \mathbb{R}$:

(i) if $x \leq y$ then $x + z \leq y + z$; and

(ii) if $0 \leq x$ and $0 \leq y$ then $0 \leq xy$.

1.5.4 Ordering $\mathbb{C}$

It is the case that $(\mathbb{C}, +, .)$ forms a field, however they are not an ordered field. One way to note this is that $i^2 = -1$, hence there are squares in the complex numbers that are negative, whereas we already noted that squares must be positive in ordered fields.

Note however that we can pre-order the complex numbers. One such pre-ordering of the complex numbers is $a + bi \preceq c + di$ when $\sqrt{a^2 + b^2} \leq \sqrt{c^2 + d^2}$ (Ie. based on their distance from the origin).

1.6 Metric Properties

Note that metric spaces are examined in detail during KMA352 Analysis 3B. For KMA315 we will attempt to avoid analysing the real numbers from such a general/abstract framework. However it certainly does not hurt to become familiar with their definition and what metric spaces examine at this stage of one's learning, and readers should note that papers at the research level do assume a much more thorough background knowledge of metric spaces and other areas of pure mathematics.

1.6.1 Definition: Given a set X, a *metric function* or *distance function* is a function $f : X \times X \to \mathbb{R}$ such that for any $x, y, z \in X$:

(i) $d(x, y) \geq 0$ (non-negativity);

(ii) $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles);

(iii) $d(x, y) = d(y, x)$ (symmetry); and

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

A pair (X, d) where X is a set and d is a metric function (on pairs of elements of X) is typically referred to as a *metric space*.

1.6.2 Examples: A few examples of metric spaces include:

(i) The real numbers with the metric function $d(a, b) = |b - a|$;

(ii) $\mathbb{R}^2$ with any of the metric functions (and there are probably more):

(I) $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Euclidean metric);

(II) $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$ (Manhatten metric);

(III) $d((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$ (Chebyshev metric).

Note that the non-negativity condition for a metric space is actually not necessary, since it follows from the other three conditions, which may be proved as follows.

1.6.3 Proposition: The non-negativity condition for a metric space follows from the other three conditions.

Proof. First suppose the other three conditions are met. Then $0 = d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y)$. Dividing both sides by 2 we have $0 \leq d(x, y)$, which is the non-negativity condition for a metric space. □

You may find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Metric_space`.

Chapter 2

Sequences

2.0.4 Definition: A *sequence of real numbers* is any function $f : \mathbb{N} \to \mathbb{R}$. Given such a sequence, we often denote $f(n)$ as f_n for all $n \in \mathbb{N}$, then denote our sequence as $(f_n)_{n=0}^{\infty}$. You may see sequences of real numbers referred to elsewhere as *real sequences* or as *sequences in* $\mathbb{R}$.

2.0.5 Example: Some examples of how you may see sequences defined are as follows:

(i) $(\frac{1}{n})_{n=0}^{\infty}$;

(ii) $(y_n)_{n=0}^{\infty}$ where $y_n = \frac{(-1)^n}{n+1}$ for all $n \in \mathbb{N}$; and

(iii) $(a_n)_{n=0}^{\infty}$ where $a_0 = 1$ and $a_{n+1} = \sqrt{3a_n}$ for all $n \in \mathbb{N}$.

2.1 Boundedness

2.1.1 Definition: Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers. It is said that:

(i) $(y_n)_{n=0}^{\infty}$ is *bounded above* if there exists $U \in \mathbb{R}$ such that $y_n \leq U$ for all $n \in \mathbb{N}$;

(ii) $(y_n)_{n=0}^{\infty}$ is *bounded below* if there exists $L \in \mathbb{R}$ such that $L \leq y_n$ for all $n \in \mathbb{N}$; and

(iii) $(y_n)_{n=0}^{\infty}$ is *bounded* if it is bounded both above and below.

2.2 Monotonicity

2.2.1 Definition: Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers. It is said that:

(i) $(y_n)_{n=0}^{\infty}$ is *monotone increasing* if $y_{n+1} > y_n$ for all $n \in \mathbb{N}$;

(ii) $(y_n)_{n=0}^{\infty}$ is *monotone non-decreasing* if $y_{n+1} \geq y_n$ for all $n \in \mathbb{N}$;

(iii) $(y_n)_{n=0}^{\infty}$ is *monotone decreasing* if $y_{n+1} < y_n$ for all $n \in \mathbb{N}$;

(iv) $(y_n)_{n=0}^{\infty}$ is *monotone non-increasing* if $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$;

(v) $(y_n)_{n=0}^{\infty}$ is *strictly monotone* if it is either monotone increasing, or decreasing; and

(vi) $(y_n)_{n=0}^{\infty}$ is *monotone* if it is either monotone increasing, non-decreasing, decreasing or non-increasing.

2.3 Convergence to limits and divergence

2.3.1 Definition: Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers. It is said that $(y_n)_{n=0}^{\infty}$ *converges to* $L \in \mathbb{R}$ and denoted $\lim_{n\to\infty} y_n$, if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|y_n - L| < \varepsilon$ for all $n \geq N$.

You may find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Limit_of_a_sequence`.

2.4 Operations

2.4.1 Definition: Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers, and let $c \in \mathbb{R}$.

(i) $(a_n)_{n=0}^{\infty} + (b_n)_{n=0}^{\infty} = (a_n + b_n)_{n=0}^{\infty}$;

(ii) $(a_n)_{n=0}^{\infty} - (b_n)_{n=0}^{\infty} = (a_n - b_n)_{n=0}^{\infty}$;

(iii) $(a_n)_{n=0}^{\infty}(b_n)_{n=0}^{\infty} = (a_nb_n)_{n=0}^{\infty}$;

(iv) $\frac{(a_n)_{n=0}^{\infty}}{(b_n)_{n=0}^{\infty}} = (\frac{a_n}{b_n})_{n=0}^{\infty}$; and

(v) $c(a_n)_{n=0}^{\infty} = (ca_n)_{n=0}^{\infty}$;

2.4.2 Proposition: If $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are bounded above, then $(a_n + b_n)_{n=0}^{\infty}$ is also bounded above.

Proof. Since $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are bounded above, there exists $U_a, U_b \in \mathbb{R}$ such that $a_n < U_a$ and $b_n < U_b$ for all $n \in \mathbb{N}$. It trivially follows that $a_n + b_n < U_a + U_b$ for all $n \in \mathbb{N}$ and hence that $U_a + U_b$ is an upper bound for $(a_n + b_n)_{n=0}^{\infty}$. Therefore $(a_n + b_n)_{n=0}^{\infty}$ is bounded above. $\square$

2.5 Limit theorems

2.5.1 Lemma: If a sequence of real numbers is monotone non-decreasing and bounded above then it converges to its supremum (least upper bound).

Proof. Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers that is monotone increasing and bounded above. Since $\{y_n : n \in \mathbb{N}\}$ has an upper bound, it follows from the Least-Upper-Bound Property that $\{y_n : n \in \mathbb{N}\}$ has a least upper bound in the real numbers, which we shall denote by u, Ie. $\sup_{n\in\mathbb{N}} y_n = u$.

Now, for each $\varepsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that $y_N > u - \varepsilon$, since otherwise $u - \varepsilon$ would also be an upper bound, which would contradict u being the <u>least</u> upper bound. Furthermore, since $y_n \geq y_N$ for all $n > N$ (which follows from $(y_n)_{n=0}^{\infty}$ being monotone non-decreasing), we have $y_n > u - \varepsilon$ for all $n \geq N$. Rearranging we get $|u - y_n| < \varepsilon$ for all $n \geq N$, which establishes that $(y_n)_{n=0}^{\infty}$ converges to u. $\square$

2.5.2 Lemma: If a sequence of real numbers is monotone non-increasing and bounded below then it converges to its infimum (greatest lower bound).

Proof. Assignment 1 Question 5. $\square$

2.5.3 Theorem: (**Squeeze theorem**) Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ be sequences of real numbers. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ then $\lim_{n\to\infty} b_n = L$.

Proof. It follows from $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ that for each $\varepsilon > 0$ there exists $N_a, N_c \in \mathbb{N}$ such that: $|a_n - L| < \varepsilon$ for all $n \geq N_a$; and $|c_n - L| < \varepsilon$ for all $n \geq N_c$. Let $N = \max\{N_a, N_c\}$, then for each $n \geq N$, $-\varepsilon < a_n - L < b_n - L < c_n - L < \varepsilon$. Hence $|b_n - L| < \varepsilon$ for all $n \geq N$, ie. $\lim_{n\to\infty} b_n = L$. $\square$

2.5.4 Theorem: A monotone sequence of real numbers converges if and only if the sequence is bounded.

Proof. First let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers that converges to $L \in \mathbb{R}$. Ie. for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|y_n - L| < \varepsilon$ for all $n \geq N$. Rearranging the inequality we get $L - 1 < y_n < L + 1$ for all $n \geq N$, and hence that $(y_n)_{n=N}^{\infty}$ is bounded below by $L - 1$ and bounded above by $L+1$. It trivially follows that $\min\{y_1, \ldots, y_N, L-1\}$ is a lower bound for $(y_n)_{n=0}^{\infty}$ and that $\max\{y_1, \ldots, y_N, L+1\}$ is an upper bound for $(y_n)_{n=0}^{\infty}$. Since $(y_n)_{n=0}^{\infty}$ is any convergent sequence of real numbers, we have established that every convergent sequence of real numbers is bounded (note that we did not make any mention of monotonicity for the first half of the proof).

Conversely, let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers that is monotone and bounded. We already established in Lemma 2.5.1 that if $(y_n)_{n=0}^{\infty}$ is monotone non-decreasing then it converges, and in Lemma 2.5.2 that if $(y_n)_{n=0}^{\infty}$ is monotone non-increasing that it also converges. $\square$

You may find it useful/insightful/interesting to read
`https://en.wikibooks.org/wiki/Real_Analysis/Sequences`.

2.6 Cauchy sequences

2.6.1 Definition: A sequence of real numbers $(y_n)_{n=0}^{\infty}$ is referred to as *a Cauchy sequence* if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|y_n - y_m| < \varepsilon$ for all $m, n \geq N$.

You may find it useful/insightful/interesting to read

`https://en.wikipedia.org/wiki/Cauchy_sequence`.

2.7 Completeness of the real numbers

2.7.1 Definition: A metric space in which every Cauchy sequence converges is typically referred to as *complete*.

You may find it useful/insightful/interesting to read:

(i) `https://en.wikipedia.org/wiki/Completeness_of_the_real_numbers`; and

(ii) `https://en.wikipedia.org/wiki/Complete_metric_space`.

2.8 Subsequences

2.8.1 Definition: Given a sequence of real numbers $(y_n)_{n=0}^{\infty}$ and a strictly increasing sequence of natural numbers $n_1 < n_2 < \ldots$, we typically refer to $(y_{n_k})_{k=0}^{\infty}$ as a *subsequence of* $(y_n)_{n=0}^{\infty}$.

2.8.2 Lemma: Every sequence of real numbers has a monotone subsequence.

Proof. Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers. We shall refer to a positive integer $m \in \mathbb{Z}_+$ as *a peak of* $(y_n)_{n=0}^{\infty}$ if $y_m > y_n$ for all $n > m$ (ie. if y_m is greater than every proceeding term of the sequence).

If $(y_n)_{n=0}^{\infty}$ has infinitely many peaks $n_0 < n_1 < \ldots < n_k < \ldots$ then the subsequence $(y_{n_k})_{k=0}^{\infty}$ corresponding to the peaks is monotonically decreasing and we are done.

Suppose $(y_n)_{n=0}^{\infty}$ has $k \in \mathbb{Z}_{\geq 0}$ peaks. Let N be the last such peak (set $N = 0$ if $(y_n)_{n=0}^{\infty}$ does not have any peaks) and set $n_1 = N + 1$. Since n_1 is not a peak of $(y_n)_{n=0}^{\infty}$, there exists $n_2 > n_1$ such that $y_{n_2} \geq y_{n_1}$. Again, since n_2 is not a peak of $(y_n)_{n=0}^{\infty}$, there exists $n_3 > n_2$ such that $y_{n_3} \geq y_{n_2}$. Repeating this process gives us a monotone non-decreasing subsequence $(y_{n_k})_{k=0}^{\infty}$. $\square$

2.9 Cluster points of sequences

2.9.1 Definition: Let $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers. A point $x \in \mathbb{R}$ is referred to as *a cluster point* (or as *an accumulation point*) of $(y_n)_{n=0}^{\infty}$ if there is a subsequence $(y_{n_k})_{k=0}^{\infty}$ that converges to x, ie. $\lim_{k\to\infty} y_{n_k} = x$.

You may find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Limit_point#Types_of_limit_points`.

2.10 Bolzano-Weierstrass theorem

2.10.1 Theorem: (**Bolzano-Weierstrass theorem**) Each bounded sequence of real numbers has a convergent subsequence.

Proof 1

Proof. Let $(y_n)_{n=0}^{\infty}$ be a bounded sequence of real numbers. It follows from Lemma 2.8.2 that $(y_n)_{n=0}^{\infty}$ has a monotone subsequence $(y_{n_k})_{k=0}^{\infty}$ (which is also bounded). Then since $(y_{n_k})_{k=0}^{\infty}$ is bounded and monotone, it follows from Theorem 2.5.4 that $(y_{n_k})_{k=0}^{\infty}$ converges. □

Proof 2 slightly modified version of
`https://en.wikibooks.org/wiki/Real_Analysis/Sequences#Theorem_.28Bolzano.E2.80.94Weierstrass.29`

Proof. Let:

(i) $(y_n)_{n=0}^{\infty}$ be a sequence of real numbers that is bounded by $M \in \mathbb{R}_+$, ie. $|y_n| < M$ for all $n \in \mathbb{N}$;

(ii) $A = \{a \in \mathbb{R} : |a| \leq M \text{ and } a \leq y_n \text{ for infinitely many } n\}$, which is non-empty since $-M \in A$ and is trivially bounded above by M; and

(iii) $L = \sup A = \min\{b \in \mathbb{R} : b \geq a \text{ for all } a \in A\}$.

To establish that $(y_n)_{n=0}^{\infty}$ has a convergent subsequence, we first establish by contradiction that for each $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that $|y_n - L| < \varepsilon$. Suppose there exists $\varepsilon > 0$ such that there is only finitely many $n \in \mathbb{N}$ satisfying $|y_n - L| < \varepsilon$ *(note that, for any such ε, any $\varepsilon' \in (0, \varepsilon]$ also works without loss of generality (WLOG))*. There are two situations in which this may occur, either: there exist infinitely many $n \in \mathbb{N}$ such that $L \leq y_n$; or there exist only finitely many $n \in \mathbb{N}$ such that $L \leq y_n$.

Suppose there exist infinitely many $n \in \mathbb{N}$ such that $L \leq y_n$. Since only a finite number of these $n \in \mathbb{N}$ may satisfy $L \leq y_n < L + \varepsilon$:

(i) we must have $L < M$, if $L + \varepsilon > M$ then (WLOG) pick a smaller ε so that $L + \varepsilon \leq M$;

(ii) there must exist infinitely many $n \in \mathbb{N}$ such that $L + \varepsilon \leq y_n \leq M$.

Hence $L + \varepsilon \in A$, which contradicts L being an upper bound of A.

Hence it is not possible for:

(i) there to exist $\varepsilon > 0$ such that there is only finitely many $n \in \mathbb{N}$ satisfying $|y_n - L| < \varepsilon$; and

(ii) there to exist infinitely many $n \in \mathbb{N}$ such that $L \leq y_n$.

Next suppose there exist only finitely many $n \in \mathbb{N}$ such that $L \leq y_n$. It follows from the assumption we do still have, which is there only be finitely many $n \in \mathbb{N}$ satisfying $|y_n - L| < \varepsilon$, that there is also only finitely many $n \in \mathbb{N}$ such that $L - \varepsilon < y_n$, and such that $L - \frac{\varepsilon}{2} \leq y_n$. Since for each $a \in A$ there are infinitely many $n \in \mathbb{N}$ such that $a \leq y_n$, it follows that $a < L - \frac{\varepsilon}{2}$ for all $a \in A$, which contradicts L being the least upper bound of A.

Hence it is also not possible for:

(i) there to exist $\varepsilon > 0$ such that there is only finitely many $n \in \mathbb{N}$ satisfying $|y_n - L| < \varepsilon$; and

(ii) there to exist only finitely many $n \in \mathbb{N}$ such that $L \leq y_n$.

Therefore for each $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that $|y_n - L| < \varepsilon$.

Finally we establish that $(y_n)_{n=0}^{\infty}$ has a subsequence which converges to L. For each $k \in \mathbb{N}$, let $\varepsilon_k = \frac{1}{k}$ and chose $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $|y_{n_k} - L| < \varepsilon_k$. Then for each $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $\varepsilon_k < \varepsilon$ for all $k \geq K$. It follows that $|y_{n_k} - L| < \varepsilon$ for all $k \geq K$, and hence that $(y_{n_k})_{k=0}^{\infty}$ is a subsequence of $(y_n)_{n=0}^{\infty}$ that converges to L. $\square$

It is also possible to prove the Bolzano-Weierstrass theorem using the nested interval property, which we will cover in Section 3.8 of the next chapter. You may also find it useful/insightful/interesting to read `https://en.wikipedia.org/wiki/Bolzano-Weierstrass_theorem`.

Chapter 3

Subsets of $\mathbb{R}$

3.1 Neighbourhoods/balls

3.1.1 Definition: Given $x \in \mathbb{R}$ and $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) = \{r \in \mathbb{R} : |x - r| < \varepsilon\}$ is typically referred to as *the open neighbourhood/ball around x of radius ε* or more succinctly as just *an open ball around x*, and often denoted as $B_\varepsilon(x)$.

More generally for a metric space (X, d). Given $x \in X$ and $\varepsilon > 0$ then the set $\{x' \in X : d(x, x') < \varepsilon\}$ is typically referred to as *the open ball around x of radius ε* or more succinctly as just *an open ball around x*, and typically denoted as $B_\varepsilon(x)$.

If you are curious about neighbourhoods/balls more generally then see `https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)`.

3.2 Limit points of subsets

3.2.1 Definition: Given a subset $A \subseteq \mathbb{R}$, a point $x \in \mathbb{R}$ is referred to as *a limit point of A* if for each $\varepsilon > 0$ there exists $a \in A$ such that $a \neq x$ and $a \in (x - \varepsilon, x + \varepsilon)$.

3.2.2 Examples: (i) The only limit point of the subset $\{\frac{1}{n} : n \in \mathbb{N}\}$ is 0. Since for each $\varepsilon > 0$

there exists $N \in \mathbb{N}$ such that $\frac{1}{N} \neq 0$ and $\frac{1}{N} \in (-\varepsilon, \varepsilon)$, while for each $N \in \mathbb{N}$ the only point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ that is an element of $\left(\frac{1}{N} - \frac{1}{N(N+1)}, \frac{1}{N} + \frac{1}{N(N+1)}\right)$ is $\frac{1}{N}$.
Note: $\frac{1}{N(N+1)} = \frac{1}{N} - \frac{1}{N+1}$;

(ii) the limit points of $(0, 1)$ are $[0, 1]$;

(iii) there are no limit points of the integers $\mathbb{Z}$, the positive integers $\mathbb{Z}_+$, the natural numbers $\mathbb{N}$, etc.;

(iv) the limit points of the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathcal{C}(\mathbb{Q})$ are $\mathbb{R}$.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Limit_point`.

3.3 Open sets

3.3.1 Definition: A subset $A \subseteq \mathbb{R}$ is referred to as *open* if for each point $a \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(a) = (a - \varepsilon, a + \varepsilon) \subseteq A$.

Note that given $a \in \mathbb{R}$ and $\varepsilon > 0$, $B_\varepsilon(a) = \{r \in \mathbb{R} : |a - r| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$ is typically referred to as an *open* ball around a (since it is an open subset of $\mathbb{R}$).

Let (X, d) be a metric space. A subset $A \subseteq X$ is referred to as *open* if for each $a \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(a) = \{x \in X : d(a, x) < \varepsilon\} \subseteq A$.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Open_set`.

3.4 Closed sets

3.4.1 Definition: A subset $A \subseteq \mathbb{R}$ is referred to as *closed* if it contains all of its limit points.

3.4.2 Definition: Given a subset $A \subseteq \mathbb{R}$, the *closure of A* consists of A along with all of the limit points of A. The closure of A is often denoted as $cl(A)$.

Note that some authors first define the closure of subsets of $\mathbb{R}$ then define a subset of $\mathbb{R}$ to be closed when it is equal to its closure. It is not too difficult to convince oneself that both definitions are equivalent.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Closed_set`.

3.5 Clopen sets

3.5.1 Definition: A subset $A \subseteq \mathbb{R}$ is typically referred to as *clopen* if it is both open and closed.

3.5.2 Examples: The only clopen subsets of the real numbers $\mathbb{R}$ are the null set $\varnothing$ and the real numbers $\mathbb{R}$ (which may be proved, though we wont in these notes).

3.6 Complements

3.6.1 Definition: Given a subset $A \subseteq \mathbb{R}$, the set $\{r \in \mathbb{R} : r \notin A\}$ is typically referred to as *the complement of A (in* $\mathbb{R}$*)* and denoted as $\mathcal{C}(A)$.

3.6.2 Proposition: Given a subset $A \subseteq \mathbb{R}$:

(i) if A is open then $\mathcal{C}(A)$ is closed; and

(ii) if A is closed then $\mathcal{C}(A)$ is open.

Proof. (i) Suppose A is open and let l be a limit point of $\mathcal{C}(A)$. If it were the case that $l \in A$ then there would exist $\varepsilon > 0$ such that $(l-\varepsilon, l+\varepsilon) \nsubseteq A$ which would contradict A being open, hence it must be the case that each limit point of $\mathcal{C}(A)$ is contained in $\mathcal{C}(A)$, and hence that $\mathcal{C}(A)$ must be closed.

(ii) Suppose A is closed and hence contains each of its limit points. Since each $r \in \mathcal{C}(A)$ is not a limit point of A, there exists $\varepsilon > 0$ such that $(r - \varepsilon, r + \varepsilon) = B_\varepsilon(r) \subseteq \mathcal{C}(A)$, which establishes that $\mathcal{C}(A)$ is open.

□

3.7 Dense sets

3.7.1 Definition: A subset $A \subseteq \mathbb{R}$ is referred to as *dense* if every point $x \in \mathbb{R}$ is either an element of A or a limit point of A, ie. when $cl(A) = \mathbb{R}$.

3.7.2 Examples: It may be proved, though we wont in these notes, that the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathcal{C}(\mathbb{Q})$ are dense subsets of the real numbers. Which establishes that:

(i) for each $q_1, q_2 \in \mathbb{Q}$ such that $q_1 < q_2$ there exists $i \in \mathcal{C}(\mathbb{Q})$ such that $q_1 < i < q_2$; and

(ii) for each $i_1, i_2 \in \mathcal{C}(\mathbb{Q})$ such that $i_1 < i_2$ there exists $q \in \mathbb{Q}$ such that $i_1 < q < i_2$.

You should take mental note of this information about the rational numbers $\mathbb{Q}$ and $\mathcal{C}(\mathbb{Q})$, both for this course and for general knowledge as a mathematician.

The reader may be aware that a set X with an infinite number of elements (ie. $|X| = \infty$) is typically referred to as:

(i) *countable* or *countably infinite* if there exists a bijection (one-to-one mapping, aka an injective and surjective function) from X to the natural numbers; and

(ii) typically referred to as *uncountable* or *uncountably infinite* otherwise.

The reader may also be aware that the rational numbers $\mathbb{Q}$ are countable while the irrational numbers are uncountable. The author feels this is one place where common intuition for the terminology breaks down with what is actually the case when one properly takes into account the precise definitions of terms being used.

To highlight what I mean, we say that there is an uncountably infinite number of irrational numbers and that there is only a countably infinite number of rational numbers, yet there exists rational numbers between any two irrational numbers. Without pre-emptively clarifying what is meant by *more* then it is ambiguous to the reader/listener what *more* even means.

The author typically lumps these people into the same group as those who say that $1+2+3+\ldots = -\frac{1}{12}$ without explaining to their reader/listener that for the particular context, the expression $1+2+3+\ldots$ is <u>not</u> actually expressing what most people, especially non-mathematicians, would reasonably assume it does. It is plausible that people who do this are aware of what they are doing, which the majority will claim if you ever call them out on it, however at least some of those people are simply being defensive at being called out.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Dense_set`.

3.8 Nested interval property

Recall from Definition 1.2.1 that an interval of $\mathbb{R}$ is a subset $I \subseteq \mathbb{R}$ such that for each $a, b \in I$, $(a, b) \subseteq I$.

3.8.1 Definition: An interval $I \subseteq \mathbb{R}$ is typically referred to as:

(i) *an open interval (of* $\mathbb{R}$*)* if it is an open subset of $\mathbb{R}$; and

(ii) *a closed interval (of* $\mathbb{R}$*)* if it is a closed subset of $\mathbb{R}$.

3.8.2 Definition: Let I_1 and I_2 be intervals of $\mathbb{R}$. When $I_2 \subseteq I_1$ it is said that I_2 *is nested inside* I_1. A sequence $(I_n)_{n=0}^{\infty}$ of intervals of $\mathbb{R}$ is referred to as *a sequence of nested intervals (of* $\mathbb{R}$*)* if I_{n+1} is nested inside I_n (ie. $I_{n+1} \subseteq I_n$) for all $n \in \mathbb{N}$.

3.8.3 Theorem: (Nested Interval Property) Let $(I_n)_{n=0}^{\infty}$ be a sequence of nested intervals. If I_n is closed and bounded for all $n \in \mathbb{N}$ then $\bigcap_{n=0}^{\infty} I_n$ is non-empty.

Proof. For each $n \in \mathbb{N}$, it trivially follows from I_n being closed and bounded that I_n has finite end-points which are both included, hence there exists $a_n, b_n \in \mathbb{R}$ such that $a_n \leq b_n$ and $I_n = [a_n, b_n]$, ie. $(I_n)_{n=0}^{\infty} = ([a_n, b_n])_{n=0}^{\infty}$.

Now it follows from I_{n+1} being nested inside I_n for all $n \in \mathbb{N}$ that for each $m \in \mathbb{N}$, b_m is an upper bound of $(a_n)_{n=0}^{\infty}$. Therefore by the supremum property of $\mathbb{R}$ (Property 1.3.3 in these notes), $\sup_{n\in\mathbb{N}} a_n$ exists, and since it is the least upper bound we have $a_m \leq \sup_{n\in\mathbb{N}} a_n \leq b_m$ for all $m \in \mathbb{N}$. Therefore $\sup_{n\in\mathbb{N}} a_n \in I_m$ for all $m \in \mathbb{N}$, which establishes that $\sup_{n\in\mathbb{N}} a_n \in \bigcap_{n=0}^{\infty} I_n$ and hence that $\bigcap_{n=0}^{\infty} I_n$ is non-empty. $\square$

For a sequence $([a_n, b_n])_{n=0}^{\infty}$ of nested closed and bounded intervals of $\mathbb{R}$, one can actually show that

$$\bigcap_{n=0}^{\infty} [a_n, b_n] = \left[\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right] = \left[\sup_{n\in\mathbb{N}} a_n, \inf_{n\in\mathbb{N}} b_n\right].$$

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Nested_intervals`.

3.9 Covers of subsets

3.9.1 Definition: Given:

(i) $A \subseteq \mathbb{R}$; and

(ii) for each $n \in \mathbb{N}$, $C_n \subseteq \mathbb{R}$.

$\{C_n : n \in \mathbb{N}\}$ is typically referred to as:

(i) *a cover of* A if $A \subseteq \bigcup_{n=0}^{\infty} C_n$; and

(ii) *an open cover of* A if it is a cover of A and C_n is open for all $n \in \mathbb{N}$.

Furthermore if there exists $M \in \mathbb{N}$ such that $A \subseteq \bigcup_{n=0}^{M} C_n$ then $\Big\{C_n : n \in \{0, \ldots, M\}\Big\}$ is typically referred to as *a finite subcover of* $\{C_n : n \in \mathbb{N}\}$.

3.10 Compact sets

3.10.1 Definition: A subset $A \subseteq \mathbb{R}$ is referred to as *compact* if each of its open covers has a finite subcover.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Compact_space`.

3.11 Heine-Borel theorem

3.11.1 Theorem: (**Heine-Borel theorem**) A subset $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. (slightly rephrased version of the proof on pages $334-336$ in [1]) Suppose K is compact. Since $B_n(0) = (-n, n)$ is open for all $n \in \mathbb{N}$ and $K \subseteq \bigcup_{n=0}^{\infty} B_n(0) = \mathbb{R}$, it follows that $\{B_n(0) : n \in \mathbb{N}\}$ is an open cover of K. Furthermore, it follows from K being compact that $\{B_n(0) : n \in \mathbb{N}\}$ has a finite subcover $\Big($ie. there exists $M \in \mathbb{N}$ such that $K \subseteq \bigcup_{n=0}^{M} B_n(0) = B_M(0) = (-M, M)\Big)$, and hence that K is bounded.

To establish that K is closed we will establish that the complement $\mathcal{C}(K)$ of K (ie. $\{x \in \mathbb{R} : x \notin K\}$) is open. Let $u \in \mathcal{C}(K)$ and for each $n \in \mathbb{Z}_+$ let $G_n = \{y \in \mathbb{R} : |y-u| > \frac{1}{n}\} = (-\infty, u-\frac{1}{n}) \cup (u+\frac{1}{n}, \infty)$, which is trivially open and satisfies $\mathbb{R} - \{u\} = \bigcup_{n=1}^{\infty} G_n$.

It follows from $u \notin K$ that $K \subseteq \bigcup_{n=1}^{\infty} G_n$, and further follows from K being compact that there exists $M \in \mathbb{N}$ such that $K \subseteq \bigcup_{n=1}^{M} G_n = G_M = (\infty, u-\frac{1}{M}) \cup (u+\frac{1}{M}, \infty)$. From which it furthermore follows that $K \cap (u-\frac{1}{M}, u+\frac{1}{M})$ is empty, and hence that the interval $(u-\frac{1}{M}, u+\frac{1}{M}) \subseteq \mathcal{C}(K)$. Since u was any arbitrary point of $\mathcal{C}(K)$ we have established that $\mathcal{C}(K)$ is open, and hence completed this half of the proof.

Conversely, suppose K is closed and bounded *(Assumption 1)*. Let $(A_n)_{n=0}^{\infty}$ be an open cover of K (ie. $K \subseteq \bigcup_{n=0}^{\infty} A_n$ and A_n is open for all $n \in \mathbb{N}$), and suppose $(A_n)_{n=0}^{\infty}$ has no finite subcover of K

(Assumption 2).

Part of Assumption 1 is that K is bounded, hence there exists $M \in \mathbb{R}_+$ such that $K \subseteq [-M, M]$. Let $I_1 = [-M, M]$ and bisect I_1 into $I_1^{'} = [-M, 0]$ and $I_1^{''} = [0, M]$. At least one of $K \cap I_1^{'}$ and $K \cap I_1^{''}$ must be:

(i) non-empty (since $K \subseteq I_1 = I_1^{'} \cup I_1^{''}$); and

(ii) not be contained in the union of a finite number of sets from $(A_n)_{n=0}^{\infty}$ (since if both $K \cap I_1^{'}$ and $K \cap I_1^{''}$ are contained in the union of a finite number of sets from $(A_n)_{n=0}^{\infty}$ then $K = (K \cap I_1^{'}) \bigcup (K \cap I_1^{''})$ would also be contained in the union of a finite number of sets from $(A_n)_{n=0}^{\infty}$, which would contradict Assumption 2).

Let $I_2 = \begin{cases} I_1^{'} & \text{if } K \cap I_1^{'} \text{ is not contained in the union of a finite number of sets from } (A_n)_{n=0}^{\infty}; \\ I_1^{''} & \text{if } K \cap I_1^{''} \text{ is not contained in the union of a finite number of sets from } (A_n)_{n=0}^{\infty}, \end{cases}$
and bisect I_2 into two closed subintervals $I_2^{'}$ and $I_2^{''}$.

Let $I_3 = \begin{cases} I_2^{'} & \text{if } K \cap I_2^{'} \text{ is not contained in the union of a finite number of sets from } (A_n)_{n=0}^{\infty}; \\ I_2^{''} & \text{if } K \cap I_2^{''} \text{ is not contained in the union of a finite number of sets from } (A_n)_{n=0}^{\infty}. \end{cases}$

Continuing this process, we obtain a sequence $(I_n)_{n=1}^{\infty}$ of nested closed and bounded intervals of $\mathbb{R}$. By the Nested Interval Property 3.8.3, there exists $l \in \bigcap_{n=1}^{\infty} I_n$.

For each $m \in \mathbb{Z}_+$, since the interval I_m is not contained in the union of a finite number of sets from $(A_n)_{n=0}^{\infty}$ and each element of K must be in at least one A_n (since $(A_n)_{n=0}^{\infty}$ covers K), the interval I_m must contain infinitely many points from K, and hence l must be a limit point of K. Since it is assumed that K is closed by Assumption 1, we must have $l \in K$.

Since $(A_n)_{n=0}^{\infty}$ covers K and $l \in K$, there must exist $m \in \mathbb{N}$ such that $l \in A_m$. Since A_m is open, there exists $\varepsilon > 0$ such that $B_{\varepsilon}(l) = (l - \varepsilon, l + \varepsilon) \subseteq A_m$.

Now, since the intervals I_n are obtained by repeated bisections of $I_1 = [-M, M]$, the length of I_n is $\frac{M}{2^{n-2}}$. There trivially exists $N \in \mathbb{N}$ such that $\frac{M}{2^{n-2}} < \varepsilon$ for all $n \geq N$, where we get $I_n \subseteq (l - \varepsilon, l + \varepsilon) \subseteq A_m$ for all $n \geq N$, and hence that $K \cap I_n \subseteq A_m$ for all $n \geq N$, which contradicts our construction of $(I_n)_{n=1}^{\infty}$. It follows that it is not possible for Assumption 2 to hold, and hence that if a subset $K \subseteq \mathbb{R}$ is closed and bounded then it must be compact. $\square$

You may also find it useful/insightful/interesting to read `https://en.wikipedia.org/wiki/Heine%E2%80%93Borel_theorem`.

3.12 Sequentially compact sets

3.12.1 Definition: A subset $K \subseteq \mathbb{R}$ is referred to as *sequentially compact* if each sequence of points from A has a subsequence that converges to a point in A.

3.12.2 Proposition: A subset $K \subseteq \mathbb{R}$ is compact if and only if it is sequentially compact.

Proof. (slightly rephrased version of the proof on page 336 in [1]) Suppose K is compact and let $(y_n)_{n=0}^{\infty}$ be a sequence such that $y_n \in K$ for all $n \in \mathbb{N}$. It follows from the Heine-Borel Theorem 3.11.1 that K is bounded and hence that $(y_n)_{n=0}^{\infty}$ is also bounded. It follows from the Bolzano-Weierstrass Theorem 2.10.1 that $(y_n)_{n=0}^{\infty}$ has a convergent subsequence $(y_{n_k})_{k=0}^{\infty}$. It further follows from the Heine-Borel Theorem 3.11.1 that K is closed, and hence that $\lim_{k\to\infty} y_{n_k} \in K$, which establishes that K is sequentially compact.

To establish the converse, we will establish that if K is either not closed or not bounded, then there must exist a sequence in K that has no subsequence converging to a point of K.

(i) If K is not closed then there exists a limit point l of K such that $l \notin K$. Since l is a limit point of K, there exists a sequence $(y_n)_{n=0}^{\infty}$ of points from K such that $\lim_{n\to\infty} y_n = l \notin K$. Since every subsequence also converges to l, there is no subsequence that converges to a point in K.

(ii) If K is not bounded, then there exists a monotonically increasing sequence $(y_n)_{n=0}^{\infty}$ of points from K such that $|y_n| > n$ for all $n \in \mathbb{N}$, of which every subsequence is trivially unbounded. Hence $(y_n)_{n=0}^{\infty}$ has no convergent subsequences (note $(y_n)_{n=0}^{\infty}$ need not be monotonically increasing, however we can always choose such a sequence which is and it is easier to convince yourself that every subsequence is unbounded when it is).

$\square$

While we will not cover it in this unit, it turns out that for any metric space (X, d), the notions of compactness and sequential compactness are equivalent. While more generally for topological spaces, the notions of compactness and sequential compactness are <u>not</u> equivalent.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Sequentially_compact_space`.

Chapter 4

Real-valued continuous functions on $\mathbb{R}$

4.0.3 Definition: Given two sets X and Y, a *Y-valued function on X*, often just referred to as a *function*, is a mapping from each element of X to an element of Y. Such a function, say f, will often be written as $f : X \to Y$, and given $x \in X$ then $f(x)$ typically denotes the element of Y that f maps x to.

Furthermore given $Z \subseteq X$, the set $\{f(z) : z \in Z\}$ is often denoted as $f(Z)$.

4.0.4 Definition: Let:

(i) X and Y be sets; and

(ii) $f : X \to Y$ be a Y-valued function on X.

It is typically said that f is:

(i) *injective* when for each $x, x' \in X$, $f(x) = f(x')$ implies $x = x'$;

(ii) *surjective* when $f(X) = Y$, ie. for all $y \in Y$ there exists $x \in X$ such that $f(x) = y$; and

(iii) *bijective* when it is both injective and surjective.

When f is bijective then we may define its *inverse*, which is typically denoted as f^{-1}, where for each $y \in Y$, $f^{-1}(y)$ is the unique $x \in X$ such that $f(x) = y$.

4.1 Limits of functions

4.1.1 Definition: Let:

(i) $A \subseteq \mathbb{R}$;

(ii) $f : A \to \mathbb{R}$ be a real-valued function on A; and

(iii) $p, L \in \mathbb{R}$.

It is said that:

(i) *the limit of f as x approaches p from the left is equal to L*, and denoted as $\lim_{x\to p^+} f(x) = L$, if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for each $x \in (p, p + \varepsilon)$, $|f(x) - L| < \delta$;

(ii) *the limit of f as x approaches p from the right is equal to L*, and denoted as $\lim_{x\to p^-} f(x) = L$, if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for each $x \in (p - \varepsilon, p)$, $|f(x) - L| < \delta$;

(iii) *the limit of f as x approaches p is equal to L*, and denoted as $\lim_{x\to p} f(x) = L$, if for every $\delta > 0$ there exists $\varepsilon > 0$ such that for each $x \in (p - \varepsilon, p) \cup (p, p + \varepsilon)$, $|f(x) - L| < \delta$;

(iv) *the limit of f as x approaches ∞*, and denoted as $\lim_{x\to\infty} f(x) = L$, if for every $\delta > 0$ there exists $N > 0$ such that for each $x \geq N$, $|f(x) - L| < \delta$; and

(v) *the limit of f as x approaches $-\infty$*, and denoted as $\lim_{x\to-\infty} f(x) = L$, if for every $\delta > 0$ there exists $N > 0$ such that for each $x \leq N$, $|f(x) - L| < \delta$.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Limit_of_a_function`.

4.2 Continuous functions

4.2.1 Definition: Let:

(i) $A \subseteq \mathbb{R}$;

(ii) $a \in A$; and

(iii) $f : A \to \mathbb{R}$ be a real-valued function on A.

Such a function f is referred to:

(i) as *continuous at the point* a if for each $\delta > 0$ there exists $\varepsilon > 0$ such that for all $x \in B_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$, $f(x) \in B_\delta(f(a)) = (f(a)-\delta, f(a)+\delta)$ $\Big($ie. $f(B_\varepsilon(a)) \subseteq B_\delta(f(a))$ $\Big)$; and

(ii) as *continuous on* A if f is continuous at every point in A.

4.2.2 Examples: (i) $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathcal{C}(\mathbb{Q}) \end{cases}$ is continuous at $x = 0$ and discontinuous everywhere else.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Continuous_function`.

4.3 Sequences of functions

4.3.1 Pointwise convergence

4.3.1 Definition: Given $A \subseteq \mathbb{R}$ and a sequence $(f_n)_{n=0}^\infty$ of real-valued functions on A, it is said that $(f_n)_{n=0}^\infty$ *converges pointwise to* $f : A \to \mathbb{R}$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in A$.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Pointwise_convergence`.

4.3.2 Example: For each $n \in \mathbb{N}$, let $f_n(x) = x + \frac{1}{n}$ for all $x \in \mathbb{R}$. For each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}(x + \frac{1}{n}) = x$. Hence the pointwise limit of $(f_n)_{n=0}^{\infty}$ is $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$ for all $x \in \mathbb{R}$.

4.3.3 Example: For each $n \in \mathbb{N}$, let $f_n(x) = \frac{x}{n}$ for all $x \in \mathbb{R}$. For each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0$. Hence the pointwise limit of $(f_n)_{n=0}^{\infty}$ is $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for all $x \in \mathbb{R}$.

4.3.2 Uniform convergence

4.3.4 Definition: Given $A \subseteq \mathbb{R}$ and a sequence $(f_n)_{n=0}^{\infty}$ of real-valued functions on A, it is said that $(f_n)_{n=0}^{\infty}$ *converges uniformly to* $f : A \to \mathbb{R}$ if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in A$.

4.3.5 Example: For each $n \in \mathbb{N}$, let $f_n(x) = x + \frac{1}{n}$ for all $x \in \mathbb{R}$. For each $\varepsilon > 0$, for all $n > \frac{1}{\varepsilon}$ we have $|f_n(x) - f(x)| = \frac{1}{n} < \varepsilon$. Hence the uniform limit of $(f_n)_{n=0}^{\infty}$ is $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$ for all $x \in \mathbb{R}$.

4.3.6 Example: For each $n \in \mathbb{N}$, let $f_n(x) = \frac{x}{n}$ for all $x \in \mathbb{R}$. For each $\varepsilon > 0$ and all $n \in \mathbb{N}$, we trivially have $f_n(x) = \frac{x}{n} > \varepsilon$ for all $x > n\varepsilon$. Hence $(f_n)_{n=0}^{\infty}$ does not converge uniformly. What if we restrict each f_n to the domain $[a, b]$ where $a, b \in \mathbb{R}$ such that $a < b$?

4.3.7 Example: For each $n \in \mathbb{Z}_+$, let $f_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ nx^2 & \text{if } 0 \leq x < \frac{1}{\sqrt{n}} \\ 1 & \text{if } x \geq \frac{1}{\sqrt{n}} \end{cases}$.

4.3.8 Proposition: If $(f_n)_{n=0}^{\infty}$ converges uniformly to $f : \mathbb{R} \to \mathbb{R}$ then $(f_n)_{n=0}^{\infty}$ also converges pointwise to f.

Proof. Since $(f_n)_{n=0}^{\infty}$ converges uniformly to f, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in \mathbb{R}$. Ie. for each $x \in \mathbb{R}$, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$, which establishes that $(f_n(x))_{n=0}^{\infty}$ converges to $f(x)$ for all $x \in \mathbb{R}$, and hence that $(f_n)_{n=0}^{\infty}$ converges pointwise to f. $\square$

You may also find it useful/insightful/interesting to read `https://en.wikipedia.org/wiki/Uniform_convergence`.

4.3.3 Uniform norm

4.3.9 Definition: Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a bounded real-valued function on A. *The uniform norm of f on A* typically refers to $\sup\{|f(x)| : x \in A\}$, and is often denoted as $\|f\|_A$.

4.3.10 Theorem: A sequence $(f_n)_{n=0}^{\infty}$ of bounded real-valued functions on $A \subseteq \mathbb{R}$ converges uniformly to $f : A \to \mathbb{R}$ if and only if $\lim_{n\to\infty} \|f\|_A = \lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in A\} = 0$.

Proof. (slightly rephrased version of the proof on page 245 in [1])

Suppose $(f_n)_{n=0}^{\infty}$ is a sequence of bounded real-valued functions on $A \subseteq \mathbb{R}$ converges uniformly to $f : A \to \mathbb{R}$. Therefore for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in A$. It follows that $\sup\{|f_n(x) - f(x)| : x \in A\} < \varepsilon$ for all $n \geq N$ and hence that $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in A\} = 0$.

Conversely, suppose $(f_n)_{n=0}^{\infty}$ is a sequence of bounded real-valued functions on $A \subseteq \mathbb{R}$ such that $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in A\} = 0$. Therefore for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$, $\sup\{|f_n(x) - f(x)| : x \in A\} < \varepsilon$. It follows that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and all $x \in A$, and hence that $(f_n)_{n=0}^{\infty}$ converges uniformly to f. $\square$

4.3.4 Uniform convergence theorem

The uniform convergence theorem establishes that continuous functions are closed under uniform limits (when they exist).

4.3.11 Theorem: (**Uniform Convergence Theorem**) If $(f_n)_{n=0}^{\infty}$ is a sequence of continuous real-valued functions on $A \subseteq \mathbb{R}$ that converges uniformly towards $f : A \to \mathbb{R}$, then f is also continuous.

Proof. (slightly rephrased version of the proof on page 249 in [1])

Suppose $(f_n)_{n=0}^{\infty}$ is a sequence of continuous real-valued functions on $A \subseteq \mathbb{R}$ that converges uniformly towards $f : A \to \mathbb{R}$. Let $\varepsilon > 0$, it follows from $(f_n)_{n=0}^{\infty}$ converging uniformly to f that there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq N$ and all $x \in A$;

We will establish our desired result by establishing that f must be continuous at each $a \in A$. Let $n \geq N$, it follows from f_n being continuous at a that there exists ε' such that $|f_n(x) - f_n(a)| < \frac{\varepsilon}{3}$ for all $x \in B_{\varepsilon'}(a) = (a - \varepsilon', a + \varepsilon')$.

Finally for each $x \in B_{\varepsilon'}(a)$ and $n \geq N$, by the triangle inequality we have:

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which establishes the continuity of f. $\square$

Note that if a sequence $(f_n)_{n=0}^{\infty}$ of continuous real-valued functions converges pointwise to a discontinous function f, then we may conclude that $(f_n)_{n=0}^{\infty}$ does not converge uniformly. Since if it did, it would follow from Proposition 4.3.8 that the uniform limit should also be f, which is discontinuous and would contradict the Uniform Convergence Theorem 4.3.11.

You may also find it useful/insightful/interesting to read
`https://en.wikipedia.org/wiki/Uniform_convergence#To_continuity`.

4.3.5 Interchange of limit and integral

4.3.12 Theorem: If $(f_n)_{n=0}^{\infty}$ is a sequence of real-valued functions on $[a, b] \subseteq \mathbb{R}$ that converges uniformly to $f : [a, b] \to \mathbb{R}$ then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Proof. It follows from $(f_n)_{n=0}^{\infty}$ converging uniformly to f that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$ for all $n \geq N$ and all $x \in [a, b]$.

Now, $| \int_a^b f_n(x)dx - \int_a^b f(x)dx| = | \int_a^b (f_n(x) - f(x))dx| \leq \int_a^b |f_n(x) - f(x)|dx < \int_a^b \frac{\varepsilon}{b-a}dx = \varepsilon$,

which establishes that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$. $\square$

References

[1] Robert Gardner Bartle and Donald R Sherbert. *Introduction to real analysis*, volume 2. Wiley New York, 1992.